PROPERTIES OF POSSIBLE DISPLACEMENTS FOR THEOREMS ON INTERACTION OF PARTS OF A MECHANICAL SYSTEM

(SVOISTVA VOZMOZHNYKH PEREMESHCHENII DLIA TEOREM VZAIMODEISTVIIA CHASTEI MEKHANICHESKOI SISTEMY)

PMM Vol. 31, No. 2, 1967, pp. 377-384

A. A. BOGOIAVLENSKII
(Moscow)

(Received May 30, 1966)

As we know the properties of relationships restricting the motion of a mechanical system, can be expressed in terms of the properties of possible displacements allowed by these constraints. The latter properties make it possible to construct equations of motion of mechanical systems, to find the ways of integrating them and obtaining first integrals and to draw conclusions on the mechanical properties of systems. The part played in it by fundamental theorems of dynamics and their relationship to the elementary properties of possible displacements, is well known.

Investigating less obvious properties of possible displacements related to groups of Lie's infinitesimal transformations, Poincaré [1] established equations of motion of mechanical systems based on the group theory.

In connection with this, the present paper investigates possible displacements appearing as cyclic displacements [2] for the theorems of interaction between the parts of a mechanical system, which are mathematically expressed by the first integral obtained in [3]. The latter characterises the mutual interaction of parts of a mechanical system [3] and represents an integral of cyclic displacements.

1. Let us consider a mechanical system Λ, consisting of any number of material points $m_1, m_2, \ldots$ and divided into two parts, (1) and (2). We shall use the figure given in [3], retaining its descriptive and definitive notation.

Let the origins of two rectangular coordinate systems xyz and $x'y'z'$ which are always parallel to the fixed system $x_1 y_1 z_1$, be associated with two points A and A' belonging to systems (1) and (2) respectively.

Let the coordinates of A and A' be, in terms of the fixed coordinate system α, β, γ and α', β', γ'. respectively.

Let us also denote the coordinates of centers of gravity G and G' of (1) and (2), in the fixed coordinate system, by α^0, β^0, γ^0 and $\alpha^{0\prime}$, $\beta^{0\prime}$, $\gamma^{0\prime}$ respectively

These coordinates are connected by

$$\alpha^\circ = \lambda\alpha + \alpha_0, \quad \beta^\circ = \lambda\beta + \beta_0, \quad \gamma^\circ = \lambda\gamma + \gamma_0 \qquad (1.1)$$
$$\alpha^{\circ\prime} = \lambda'\alpha' + \alpha_0', \quad \beta^{\circ\prime} = \lambda'\beta' + \beta_0', \quad \gamma^{\circ\prime} = \lambda'\gamma' + \gamma_0'$$

where α_0, β_0, γ_0, α_0', β_0' and γ_0' are arbitrary constants.

Smooth constraints imposed on the sytem allow possible helical displacements of (1) and (2) in the manner of rigid bodies. Moreover, rotations ω_1 and ω_1' are possible, directed along two moving straight lines of constant direction passing through A and A'.

We shall denote by Π a plane passing through a point C situated at the intersection of two straight lines of constant direction AB and $A'B'$, and parallel to vectors ω_1 and ω_1'.

Points B and B' have fixed locations in coordinate sytems xyz and $x'y'z'$, and their respective coordinates are a, b, c, and a', b', c'.

Let e and e′) be two unit vectors in the Π-plane passing through C, parallel to ω_1 and ω_1' respectively and let their direction cosines be l_0, m_0, n_0 and l_0', m_0', n_0'. Let now μ be the ratio $AC:AB$, μ' the ratio $A'C:A'B'$ and let A and A′ be the vectors coincident with segments AB and AC.

Possible translational displacements of the systems (1) and (2) are directed along the straight lines n and n', perpendicular to the planes passing through A', ω_1' and A, ω_1. We shall denote these planes by ν' and ν.

If we assume that lines AB and $A'B'$ are of constant direction, then the existence of point C of their intersection under possible displacements of the system, imposes one restriction on magnitudes δl and $\delta l'$. To obtain this restriction we shall produce, through C a plane, perpendicular to the line of intersection of planes ν and ν'. Now, if we construct at the point C straight lines parallel to n and n', they will lie in the produced plane, and the point of intersection of ν and ν' lying on this plane and coincident with C at the instant t, will be somewhat displaced during the translational motion of δl and $\delta l'$. Let us denote by m the distance of this point from the initial point displaced along n by δl, and by m' its distance from the initial point displaced along n' by $\delta l'$, and the angle between the planes ν and ν', by φ^*. Then, we have

$$m = m'\cos\varphi^* + \delta l \sin\varphi^*, \quad m' = m\cos\varphi^* + \delta l' \sin\varphi^* \tag{1.2}$$

from which

$$m = \frac{\delta l + \delta l' \cos\varphi^*}{\sin\varphi^*}, \qquad m' = \frac{\delta l' + \delta l \cos\varphi^*}{\sin\varphi^*} \tag{1.3}$$

follows.

Finally, we shall denote by λ and λ' the corresponding angles between the line of intersection of ν and ν' and lines CA and CA'.

When mechanical systems (1) and (2) are displaced by δl and $\delta l'$ respectively, then point C exists only if

$$m \cos\lambda = m' \cos\lambda'$$

holds. This, by (1.3), implies that

$$\delta l (\cos\lambda - \cos\lambda' \cos\varphi^*) = \delta l' (\cos\lambda' - \cos\lambda \cos\varphi^*) \tag{1.4}$$

Let us choose possible helical displacements such, that (1.4) and

$$\delta \mathbf{l} = \chi'\mu'\mathbf{e}' \times \mathbf{A}', \quad \delta \mathbf{l}' = \chi\mu\mathbf{e} \times \mathbf{A}, \quad \omega_1' = K\omega_1 \tag{1.5}$$

$$\chi'\mu' = \varkappa'\omega_1, \quad \chi\mu = \varkappa\omega_1 \quad (K, \varkappa, \varkappa' = \text{const})$$

hold.

Systems (1) and (2) are under the action of arbitrary internal forces. Action exerted by (1) and (2), can be reduced to: reaction $\mathbf{R}$ and a couple with the moment $\mathbf{H}$ at the point A, reaction $\mathbf{R}$ and a couple with the moment $\mathbf{H}'$, at the point A' and reaction $\mathbf{R}$ plus a couple with the moment $\mathbf{W}$. at the point C - These forces are such, that conditions

$$\mathbf{W}\cdot\mathbf{e} = 0, \qquad \mathbf{W}\cdot\mathbf{e}' = 0 \tag{1.6}$$

$$\omega_1 (\varkappa - \mu)\, \mathbf{eAR} = \omega_1' (\varkappa' - \mu')\, \mathbf{e}'\mathbf{A}'\mathbf{R}$$

hold

External forces acting on the systems (1) and (2) under the assumption of invariability of the systems, can be reduced to forces $\mathbf{F}$ and $\mathbf{F}'$ and couples with moments $\mathbf{M}^\circ$ and $\mathbf{M}^{\circ\prime}$

at the points A and A'.

These forces are such, that the conditions

$$\mathbf{F}\omega_1'\mathbf{A}' = 0, \qquad \mathbf{F}'\omega_1\mathbf{A} = 0, \qquad \omega_1\cdot\mathbf{M}^\circ + \omega_1'\cdot\mathbf{M}^{\circ\prime} = 0 \tag{1.7}$$

are fulfilled.

Using first two conditions of (1, 5) and first two Eqs. of (1, 7), we can write

$$\delta\alpha\Sigma X + \delta\beta\Sigma Y + \delta\gamma\Sigma Z = 0, \quad \delta\alpha'\Sigma' X + \delta\beta'\Sigma' Y + \delta\gamma'\Sigma' Z = 0$$

Here and in the following Σ denotes summation over the points of (1), while Σ'- over the points of (2).

Let us now take rectangular coordinate systems $\xi\eta\zeta$ and $\xi'\eta'\zeta'$ with origins at A and A' respectively. They are rigidly connected with solid bodies formed from the points of the systems (1) and (2) under the assumption of their invariability within those systems.

Positions of $\xi\eta\zeta$ and $\xi'\eta'\zeta'$systems with respect to xyz and $x'y'z'$ systems are defined by respective Eulerian angles ϑ, ψ, φ and $\vartheta', \psi', \varphi'$

We shall denote by u, v, w and u', v', w' the projections on the xyz and $x'y'z'$ axes of respective velocities relative to $\xi\eta\zeta$ and $\xi'\eta'\zeta'$ systems of the points of (1) and (2), while by p_1, q_1, r_1 and p_1', q_1', r_1' we shall denote the projections, on $(x, y, z; x', y', z')$ -axes of instantaneous angular velocities of mechanical systems (1) and (2) taken as rigid bodies associated with $\xi\eta\zeta$ and $\xi'\eta'\zeta'$ coordinate systems.

Rates of change of Eulerian angles are related to angular velocities given above, by

$$\frac{d\vartheta}{dt} = p_1\cos\psi + q_1\sin\psi, \qquad \frac{d\varphi}{dt} = \frac{1}{\sin\vartheta}(p_1\sin\psi - q_1\cos\psi)$$

$$\frac{d\psi}{dt} = r_1 - \cot\vartheta\,(p_1\sin\psi - q_1\cos\psi) \tag{1.8}$$

Another set of formulas referring to $x'y'z'$ and $\xi'\eta'\zeta'$, coordinates is obtained from (1, 8) by supplementing all the magnitudes except t, with primes.

Let us now denote the direction cosines between the xyz and ξ, η, ζ axes by α_i^k $(i, k = 1, 2, 3)$ where the subscripts refer to xyz-axes, and the superscripts refer to ξ, η, ζ -axes. Similarly and retaining the same sequential order of indices, we denote the direction cosines between the $x'y'z'$ and ξ', η', ζ' axes, by β_i^k.

Projections of relative velocities of the points of (1) and (2) on the $(x, y, z; x', y', z')$ and (ξ, η, ζ and ξ', η', ζ')-axes, are connected by

$$\begin{aligned} u &= \alpha_1^1\frac{d\xi}{dt} + \alpha_1^2\frac{d\eta}{dt} + \alpha_1^3\frac{d\zeta}{dt} \qquad (uvw,\ \alpha_1^i\alpha_2^i\alpha_3^i,\ \ i = 1, 2, 3) \\ u' &= \beta_1^1\frac{d\xi'}{dt} + \beta_1^2\frac{d\eta'}{dt} + \beta_1^3\frac{d\zeta'}{dt} \qquad (u'v'w',\ \beta_1^i\beta_2^i\beta_3^i,\ \ i = 1, 2, 3) \end{aligned} \tag{1.9}$$

where the symbols ($uvw\ldots$) denote cyclic permutations.

Replacing, in the above formulas, velocities with coordinates, we obtain analogous relations between the ($x, y, z, \xi, \eta, \zeta$) and ($x', y', z', \xi', \eta', \zeta'$) coordinates of the points of (1) and (2).

2. A mechanical system Λ can, at any time t, be defined in terms of Poincaré [1 and 2] variables

$$\alpha, \beta, \gamma, \vartheta, \psi, \varphi, \xi_\nu, \eta_\nu, \zeta_\nu, \alpha', \beta', \gamma', \vartheta', \psi', \varphi', \xi_\nu', \eta_\nu', \zeta_\nu' \quad (\nu = 1, 2, \ldots) \tag{2.1}$$

We shall use real independent variables

$$\frac{d\alpha}{dt},\ \frac{d\beta}{dt},\ \frac{d\gamma}{dt},\ \frac{d\alpha'}{dt},\ \frac{d\beta'}{dt},\ \frac{d\gamma'}{dt},\ p_1,\ q_1,\ r_1,\ p_1',\ q_1',\ r_1' \qquad (\eta_\alpha)$$
$$\frac{d\xi_\nu}{dt},\ \frac{d\eta_\nu}{dt},\ \frac{d\zeta_\nu}{dt},\ \frac{d\xi_\nu'}{dt},\ \frac{d\eta_\nu'}{dt},\ \frac{d\zeta_\nu'}{dt} \qquad (\eta_{i\nu}) \tag{2.2}$$
$$(\alpha = 1,\ 2,\ \ldots,\ 12;\ i = 1,\ 2,\ \ldots, 6;\ \nu = 1,\ 2.\ \ldots)$$

as parameters η_α and $\eta_{i\nu}$ of real displacement of the system.

If the system Λ consists of free points, then the magnitudes

$$\frac{d\xi_\nu}{dt},\ \frac{d\eta_\nu}{dt},\ \frac{d\zeta_\nu}{dt},\ \frac{d\xi_\nu'}{dt},\ \frac{d\eta_\nu'}{dt},\ \frac{d\zeta_\nu'}{dt} \qquad (\nu = 1,\ 2, \ldots)$$

are independent.

Constraints imposed on the system result in formation of some relationships between the above magnitudes. In that case, only independent magnitudes can be used as parameters of real displacement of the system. The above relationships do not, however, influence the reasoning which follows. Changes that may be necessary, will be perfectly obvious (e. g. in the formulas for group theory operators, which follow).

Change of the function of position $f\,(t,\ \alpha,\ \beta,\ \gamma,\ \vartheta,\ \psi,\ \varphi,\ \xi_\nu,\ \eta_\nu,\ \zeta_\nu,\ \alpha',\ \beta',\ \gamma',\ \vartheta',\ \psi',\ \varphi',\ \xi_\nu',\ \eta_\nu',\ \zeta_\nu')$ of a mechanical system over a real displacement of the system, is given by

$$df = \left\{\frac{\partial f}{\partial t} + \Sigma\eta_\alpha X_\alpha f + \Sigma\eta_{i\nu}X_{i\nu}f\right\}dt \qquad (\alpha = 1, \ldots, 12;\ i = 1, \ldots, 6;\ \nu = 1,\ 2, \ldots)$$

Operators of the infinitesimal Lie group of real displacements, are

$$X_0 = \frac{\partial}{\partial t},\quad X_1 = \frac{\partial}{\partial\alpha},\quad X_2 = \frac{\partial}{\partial\beta},\quad X_3 = \frac{\partial}{\partial\gamma},\quad X_4 = \frac{\partial}{\partial\alpha'},\quad X_5 = \frac{\partial}{\partial\beta'}$$
$$X_6 = \frac{\partial}{\partial\gamma'},\quad X_7 = \cos\psi\frac{\partial}{\partial\vartheta} - \cot\vartheta\sin\psi\frac{\partial}{\partial\psi} + \frac{\sin\psi}{\sin\vartheta}\frac{\partial}{\partial\varphi}$$
$$X_8 = \sin\psi\frac{\partial}{\partial\vartheta} + \cot\vartheta\cos\psi\frac{\partial}{\partial\psi} - \frac{\cos\psi}{\sin\vartheta}\frac{\partial}{\partial\varphi},\qquad X_9 = \frac{\partial}{\partial\psi} \tag{2.3}$$
$$X_{10} = X_7(\vartheta',\ \psi',\ \varphi'),\qquad X_{11} = X_8(\vartheta',\ \psi',\ \varphi'),\qquad X_{12} = \frac{\partial}{\partial\psi'}$$
$$X_{1\nu} = \frac{\partial}{\partial\xi_\nu},\quad X_{2\nu} = \frac{\partial}{\partial\eta_\nu},\quad X_{3\nu} = \frac{\partial}{\partial\zeta_\nu},\quad X_{4\nu} = \frac{\partial}{\partial\xi_\nu'},\quad X_{5\nu} = \frac{\partial}{\partial\eta_\nu'},\quad X_{6\nu} = \frac{\partial}{\partial\zeta_\nu'}$$
$$(\nu = 1,\ 2, \ldots)$$

An infinitesimal Lie group of real displacements contains a subgroup of possible displacements X_α, $X_{i\nu}$ $(\alpha = 1,\ 2,\ \ldots,\ 12;\ i = 1,\ 2,\ \ldots,\ 6;\ \nu = 1,\ 2,\ldots)$, which consists of two commutative subgroups of translational displacements X_1, X_2, X_3 and X_4, X_5, X_6, referred to systems (1) and (2) and possessing the following properties: $(X_i,\ X_j) = 0$, $(X_s,\ X_r) = 0$ $(i, j = 1,\ 2,\ 3;\ s,\ r = 4,\ 5,\ 6)$ of two rotational subgroups X_7, X_8, X_9 and X_{10}, X_{11}, X_{12}, referred to systems (1) and (2) and possessing the properties

$$(X_7,\ X_8) = -X_9,\quad (X_8,\ X_9) = -X_7,\quad (X_9,\ X_7) = -X_8$$
$$(X_{10},\ X_{11}) = -X_{12},\quad (X_{11},\ X_{12}) = -X_{10},\quad (X_{12},\ X_{10}) = -X_{11}$$

and of two commutative subgroups of relative displacements with the following properties

$$(X_{i\nu},\ X_{j\nu}) = 0,\quad (X_{s\nu},\ X_{r\nu}) = 0\ (i,\ j = 1,\ 2,\ 3;\ s,\ r = 4,\ 5,\ 6;\ \nu = 1,\ 2,\ldots)$$

Operators from various subgroups are mutually interchangeable.

Structural constants of the group of possible displacements are $C_{789} = C_{897} = C_{978} = C_{10,11,12} = C_{11,12,10} = C_{12,10,11} = -1$ ($C_{iks} = -C_{kis}$; $i, k = 7, 8, 9$ or $10, 11, 12$), the remaining ones are equal to zero.

Velocities of the points of (1) are given, in the fixed $x_1 y_1 z_1$-systems, by

$$\frac{dx_1}{dt} = \frac{d\alpha}{dt} + \frac{dx}{dt} = \frac{d\alpha}{dt} + u + q_1 z - r_1 y, \quad (x_1 y_1 z_1,\ \alpha\beta\gamma,\ xyz,\ uvw,\ p_1 q_1 r_1) \tag{2.4}$$

Velocities of the points of (2) in the same coordinate system are obtained from (2.4) by supplementing all the magnitudes except t, with a prime.

Kinetic energy of the system Λ is, in accordance with (1.9) and (2.4), given by

$$T = \frac{1}{2} MV_A^2 + \frac{1}{2} M'V_{A'}^2 + T_0 + T_0' + \frac{1}{2} \Sigma mV_r^2 + \frac{1}{2} \Sigma' mV_r'^2 + \\ + p_1 P + q_1 Q + r_1 R + p_1' P' + q_1' Q' + r_1' R' + \Phi + \Phi' \tag{2.5}$$

where

$$V_A^2 = \left(\frac{d\alpha}{dt}\right)^2 + \left(\frac{d\beta}{dt}\right)^2 + \left(\frac{d\gamma}{dt}\right)^2, \qquad V_{A'}^2 = \left(\frac{d\alpha'}{dt}\right)^2 + \left(\frac{d\beta'}{dt}\right)^2 + \left(\frac{d\gamma'}{dt}\right)^2$$

$$T_0 = \frac{1}{2} \{A p_1^2 + B q_1^2 + C r_1^2 - 2D q_1 r_1 - 2E r_1 p_1 - 2F p_1 q_1\}$$

$$V_r^2 = \left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2, \quad P = \frac{d\gamma}{dt} \Sigma my - \frac{d\beta}{dt} \Sigma mz + \Sigma m (yw - zv)$$

$$\Phi = \frac{d\alpha}{dt} \Sigma mu + \frac{d\beta}{dt} \Sigma mv + \frac{d\gamma}{dt} \Sigma mw$$

magnitudes Q and R are obtained from P by cyclic permutations ($\alpha\beta\gamma$, xyz, uvw), T_0', $V_r'^2$, P', Q', R' and Φ' from the corresponding unprimed functions by adding a prime to all (except t and m) of them and taking Σ' instead of Σ, while the functions

$$A,\ B,\ C,\ D,\ E,\ F,\ A',\ B',\ C',\ D',\ E',\ F',\ x,\ y,\ z,\ x',\ y',\ z',\ u,\ v,\ w,\ u',\ v',\ w' \tag{2.6}$$

are given by well known kinematic formulas [4] in defining coordinates ϑ, ψ, φ and ϑ', ψ', φ' together with the parameters

$$\frac{d\xi}{dt}, \quad \frac{d\eta}{dt}, \quad \frac{d\zeta}{dt}, \quad \frac{d\xi'}{dt}, \quad \frac{d\eta'}{dt}, \quad \frac{d\zeta'}{dt}$$

of real displacement of the system in terms of α_i^k, β_i^k and M; Also, M is the mass of system (1) while M' is the mass of system (2).

3. Although equations of motion of the systems could be written down, they are not required for our purpose.

Direct computation of auxilliary magnitudes α_i^k and β_i^k, expressed in terms of Euler's angles [4] shows, that the operators X_γ ($\gamma = 7, \ldots, 12$), applied to these magnitudes transform the tables of cosines α_i^k and β_i^k of angles between the axes, into

$$X_7(\|\alpha_i^k\|) \sim \begin{matrix} 0 & 0 & 0 \\ -\alpha_3^1 & -\alpha_3^2 & -\alpha_3^3 \\ \alpha_2^1 & \alpha_2^2 & \alpha_2^3 \end{matrix} \qquad X_8(\|\alpha_i^k\|) \sim \begin{matrix} \alpha_3^1 & \alpha_3^2 & \alpha_3^3 \\ 0 & 0 & 0 \\ -\alpha_1^1 & -\alpha_1^2 & -\alpha_1^3 \end{matrix} \tag{3.1}$$

$$X_9(\|\alpha_i^k\|) \sim \begin{matrix} -\alpha_2^1 & -\alpha_2^2 & -\alpha_2^3 \\ \alpha_1^1 & \alpha_1^2 & \alpha_1^3 \\ 0 & 0 & 0 \end{matrix} \qquad X_{10}(\|\beta_i^k\|) \sim \begin{matrix} 0 & 0 & 0 \\ -\beta_3^1 & -\beta_3^2 & -\beta_3^3 \\ \beta_2^1 & \beta_2^2 & \beta_2^3 \end{matrix}$$

$$X_{11}(\|\beta_i{}^k\|) \sim \begin{matrix} \beta_3{}^1 & \beta_3{}^2 & \beta_3{}^3 \\ 0 & 0 & 0 \\ -\beta_1{}^1 & -\beta_1{}^2 & -\beta_1{}^3 \end{matrix} \qquad X_{12}(\|\beta_i{}^k\|) \sim \begin{matrix} -\beta_2{}^1 & -\beta_2{}^2 & -\beta_2{}^3 \\ \beta_1{}^1 & \beta_1{}^2 & \beta_1{}^3 \\ 0 & 0 & 0 \end{matrix}$$

from which we can find the variation of functions (2.6) over the possible displacements X_γ ($\gamma = 7,\ldots, 12$). E.g. we have

$$X_7(x) = 0,\ X_7(y) = -z,\ X_7(z) = y,\ X_7(u) = 0,\ X_7(v) = -w,\ X_7(w) = v, \ldots$$

from which we easily find

$$X_7(A) = 0,\quad X_7(B) = 2D,\quad X_7(C) = -2D,\quad X_7(D) = C - B,\ X_7(E) = F, \ldots$$

In xyz and $x'y'z'$ coordinate systems we have

$$\Sigma mx = M\,[(\lambda - 1)\alpha + \alpha_0] \quad (xyz,\ \alpha\beta\gamma,\ \alpha_0\beta_0\gamma_0) \tag{3.2}$$
$$\Sigma' mx' = M'\,[(\lambda' - 1)\alpha' + \alpha_0'] \ (x'y'z',\ \alpha'\beta'\gamma',\ \alpha_0'\beta_0'\gamma_0')$$

Expression (2.5) for kinetic energy yields, by (2.4) and (3.2),

$$\frac{\partial T}{\partial\left(\dfrac{d\alpha}{dt}\right)} = \lambda M \frac{d\alpha}{dt} \quad (\alpha\beta\gamma) \qquad \frac{\partial T}{\partial\left(\dfrac{d\alpha'}{dt}\right)} = \lambda' M' \frac{d\alpha'}{dt} \quad (\alpha'\beta'\gamma') \tag{3.3}$$

$$\frac{\partial T}{\partial p_1} = \frac{d\gamma}{dt}\Sigma my - \frac{d\beta}{dt}\Sigma mz + \Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) = S_1 \qquad (p_1q_1r_1,\ \alpha\beta\gamma,\ xyz,\ S_1S_2S_3)$$

$$\frac{\partial T}{\partial p_1'} = \frac{d\gamma'}{dt}\Sigma' my' - \frac{d\beta'}{dt}\Sigma' mz' + \Sigma' m\left(y'\frac{dz'}{dt} - z'\frac{dy'}{dt}\right) = S_1' \ (p_1'q_1'r_1',\ \alpha'\beta'\gamma',\ x'y'z',\ S_1'S_2'S_3')$$

For the points of system (1) we have

$$\delta x_1 = \delta\alpha + \sigma_1 z - \rho_1 y + \delta x_r,\ \pi_1 = p_1\varepsilon \ (x_1y_1z_1,\ \alpha\beta\gamma,\ \sigma_1\rho_1\pi_1,\ xyz,\ p_1q_1r_1) \tag{3.4}$$

where δx_r, δy_r and δz_r are the relative displacements of points of (1) referred to the xyz axes in terms of projections on these axes.

Formulas for the system (2) are obtained from (3.4) by indexing the appropriate magnitudes with a prime

Work done by external forces acting on the system Λ, the forces exerted by (2) on (1) and the forces exerted by (1) on (2) are given, over any possible displacement, by

$$\begin{aligned} \delta U = {} & \delta\alpha(\Sigma X - R_x) + \delta\beta(\Sigma Y - R_y) + \delta\gamma(\Sigma Z - R_z) + \pi_1(H_x - W_x) + \\ & + \sigma_1(H_y - W_y) + \rho_1(H_z - W_z) + \delta\alpha'(\Sigma' X + R_x) + \delta\beta'(\Sigma' Y + R_y) + \\ & + \delta\gamma'(\Sigma' Z + R_z) + \pi_1'(H_x' + W_x') + \sigma_1'(H_y' + W_y') + \rho_1'(H_z' + W_z') + \\ & + \Sigma X\delta x_r + \Sigma Y\delta y_r + \Sigma Z\delta z_r + \Sigma' X\delta x_r' + \Sigma' Y\delta y_r' + \Sigma' Z\delta z_r' + \delta U_r \end{aligned} \tag{3.5}$$

where δU_r is the work done by the forces listed above (without, however, the external forces), over the relative displacements $\delta x_r, \ldots, \delta z_r'$.

Variation of U over a possible displacement is given by $\delta U = \Sigma\omega_\alpha X_\alpha(U)$.

Here ω_α are the parameters of possible displacements. Comparing it with (3.5), we obtain

$$X_1(U) = \Sigma X - R_x,\quad X_2(U) = \Sigma Y - R_y,\quad X_3(U) = \Sigma Z - R_z$$
$$X_4(U) = \Sigma' X + R_x,\quad X_5(U) = \Sigma' Y + R_y,\quad X_6(U) = \Sigma' Z + R_z \tag{3.6}$$

$$X_7(U) = H_x - W_x, \quad X_8(U) = H_y - W_y, \quad X_9(U) = H_z - W_z$$
$$X_{10}(U) = H_x' + W_x', \quad X_{11}(U) = H_y' + W_y', \quad X_{12}(U) = H_z' + W_z'$$

If helical displacements shown above are taken as possible displacements, then the work done by internal forces is equal to zero.

4. Chetaev [2] developed a method of determination of cyclic first integrals by generalising the concept of cyclic coordinates and introduced the concept of cyclic displacements.

The latter concept can be enlarged by replacing the condition of conservation of the Lagrangian over any displacement, with the condition of conservation of the potential energy over a linear combination of possible displacements possessing constant coefficients. At the same time the condition that cyclic displacements form an Abelian subgroup of the group of possible displacements, is replaced with some structural properties of the function expressing the kinetic energy of the system.

Poincaré equations [1 and 2] for mechanical systems with k degrees of freedom under smooth holonomic constraints and acted upon by forces admitting the force function, have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \eta_i} = \sum_{\alpha,\beta} C_{\alpha i\beta}\eta_\alpha \frac{\partial L}{\partial \eta_\beta} + X_i L \qquad (i = 1, \ldots, k) \tag{4.1}$$

where $L = T + U$ is the Lagrangian function, η_i are real independent variables defining real displacements of the system, $C_{\alpha i\beta}$ are the structural constants of an intransitive group of k elements of infinitesimal operators X_i of possible displacements linear with respect to n real independent variables $x_1, \ldots, x_n$, which define the position of our mechanical system.

Displacements $X_\alpha (\alpha = s+1, \ldots, k)$ are according to Chetaev cyclic, if they satisfy the conditions

$$X_\alpha L = 0, \qquad (X_\alpha, X_\beta) = X_\alpha X_\beta - X_\beta X_\alpha = 0 \qquad (\beta = 1, \ldots, k) \tag{4.2}$$

Equations (4.1) yield, for cyclic displacements, according to (4.2), the following first integrals

$$\frac{\partial L}{\partial \eta_\alpha} = \text{const} \qquad (\alpha = s+1, \ldots, k) \tag{4.3}$$

Let the kinetic energy of the mechanical system be of the form

$$T = {}^1/_2\, \Sigma g_{\alpha\beta}\eta_\alpha\eta_\beta + \Sigma a_\alpha \eta_\alpha + T_0 \qquad (g_{\alpha\beta} = g_{\beta\alpha})$$

where the coefficients $g_{\alpha\beta}$, a_α and T_0 are the functions of variables $t, x_1, \ldots, x_n$.

Potential energy U is a function of variables $x_1, \ldots, x_n$.

Displacements X_ν can be considered as compound cyclic displacements, provided that conditions

$$\sum_{\nu=1}^{r} \lambda_\nu X_\nu (U) = 0 \ (r \leqslant k) \qquad \text{for some values of } \lambda_\nu \tag{4.4}$$

$$X_\nu (g_{\alpha\beta}) = \sum_{\gamma=1}^{k} (C_{\nu\alpha\gamma} g_{\beta\gamma} + C_{\nu\beta\gamma} g_{\alpha\gamma}), \qquad X_\nu (a_\alpha) = \sum_{\gamma=1}^{k} C_{\nu\alpha\gamma} a_\gamma$$

$$X_\nu (T_0) = 0 \qquad (\lambda_\nu = \text{const})$$

are fulfilled.

Set of the compound cyclic displacements makes it possible to obtain a first integral from equations of motion (4.1).

Multiplying the ν-th equation of (4.1) by a constant multiplier λ_ν and collecting r equations in accordance with (4.4), we obtain

$$\frac{d}{dt}\sum_{\nu=1}^{r}\lambda_\nu\frac{\partial T}{\partial\eta_\nu}=0$$

First integral for the set of compound cyclic displacements is then

$$\sum_{\nu=1}^{r}\lambda_\nu\frac{\partial T}{\partial\eta_\nu}=\text{const} \tag{4.5}$$

Displacements X_i ($i=1, \ldots, k$) can yield several sets of compound cyclic displacements X_ν and a first integral exists for each set.

If the displacement X_ν is cyclic in the Chetaev sense, then from the second condition of (4.2), we have

$$C_{\nu\beta i}=0 \qquad (\beta, i=1, \ldots, k)$$

Let us put

$$\lambda_s=0 \quad (s=1,\ldots, \nu-1, \nu+1, \ldots, r), \qquad \lambda_\nu\neq 0$$

Then (4.4) will yield

$$X_\nu(T)=0, \quad X_\nu(U)=0$$

from which the first condition of (4.2) follows.

In this case, the enlarged concept of cyclic displacement coincides with Chetaev's concept of cyclic displacements.

5. Let us assume that for the problem under consideration, the values of λ_ν are

$$\lambda_1=K\varkappa'(m'c'-n'b') \qquad (\lambda_1\lambda_2\lambda_3,\ l'm'n',\ a'b'c')$$
$$\lambda_4=\varkappa(mc-nb) \qquad (\lambda_4\lambda_5\lambda_6,\ lmn,\ abc)$$
$$\lambda_7=l,\ \lambda_8=m,\ \lambda_9=n,\ \lambda_{10}=Kl',\ \lambda_{11}=Km',\ \lambda_{12}=Kn'$$

By (1.5), (1.6) and (3.6) and assuming that

$$\mathbf{W}=\mathbf{H}-\mu\mathbf{A}\times\mathbf{R}=\mathbf{H}'-\mu'\mathbf{A}'\times\mathbf{R}$$

we have

$$\sum_{\nu=1}^{12}\lambda_\nu X_\nu(U)=0$$

Third and fourth condition of (4.4) hold, since $a_\nu=0$ and $T_0=0$.

Second condition of (4.4) can be directly verified using Expression (2.5) for potential energy and the values of structural constants.

First integral of (4.5) has the form

$$K\varkappa'\lambda M\left\{l'\left(b'\frac{d\gamma}{dt}-c'\frac{d\beta}{dt}\right)+m'\left(c'\frac{d\alpha}{dt}-a'\frac{d\gamma}{dt}\right)+n'\left(a'\frac{d\beta}{dt}-b'\frac{d\alpha}{dt}\right)\right\}+$$
$$+\varkappa\lambda'M'\left\{l\left(b\frac{d\gamma'}{dt}-c\frac{d\beta'}{dt}\right)+m\left(c\frac{d\alpha'}{dt}-a\frac{d\gamma'}{dt}\right)+n\left(a\frac{d\beta'}{dt}-b\frac{d\alpha'}{dt}\right)\right\}+lS_1+mS_2+$$
$$+nS_3+K(l'S_1'+m'S_2'+n'S_3')=\text{const}$$

Possible displacements corresponding to various operators of the groups, are considered for other problems in [5 to 7] and cyclic displacements are also given there.

BIBLIOGRAPHY

1. Poincaré H., Sur une forme nouvelle des équations de la mecanique. Compt. rend. Acad. sci. Paris, Vol. 132, p. 369-371, 1901.
2. Chetaev N. G., Ob uravneniiakh Puankaré (On the Poincaré equations). PMM, Vol. 5, No. 2, 1941.
3. Bogoiavlenskii A. A., Teoremy vzaimodeistviia chastei mekhanicheskoi sistemy (Theorems on interaction of parts of a mechanical system). PMM, Vol. 30, No. 1, 1966.
4. Appel'P., Teoreticheskaia mekhanika (Theoretical Mechanics). Vol. 2, Fizmatgiz, 1960.
5. Aminov M. Sh., Postroenie grupp vozmozhnykh peremeshchenii (Construction of groups of possible displacements). Tr. Mezhvuzovsk. konf. po prikl. teorii ustoichivosti dvizheniia i analit. mekh. (Coll. of intercollegiate conference on applied theory of stability and analytical mechanics). Kazan. aviats. Inst. pp. 21-30 1964.
6. Bogoiavlenskii A. A., Tsiklicheskie peremeshcheniia dlia obobshchennogo integrala ploshchadei (Cyclic displacements for generalized surface integrals). PMM, Vol. 25, No. 4, 1961.
7. Bogoiavlenskii A. A., Obobshchennye tsiklicheskie peremeshcheniia dlia giroskopa v kardanovom podvese v chastnom sluchae dvizheniia (Generalized cyclic displacements for a gyroscope mounted on gimbals in a particular case of motion). Tr. Mezhvuzovsk. konf. po prikl. teorii ustoichivosti dvizheniia i analit. mekh. (Coll. of intercollegiate conference on applied theory of stability and analytical mechanics). Kazan. aviats. Inst., pp. 38-44, 1964.

Translated by L. K.